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Hidden fine tuning in the quark sector of little higgs models

Benjamín Grinstein, Randall Kelley and Patipan Uttayarat

University of California, Department of Physics, La Jolla, San Diego CA 92093-0315, U.S.A.

E-mail: bgrinstein@ucsd.edu, randallkelley@physics.ucsd.edu, puttayarat@physics.ucsd.edu

Abstract: In Little Higgs models a collective symmetry prevents the higgs from acquiring a quadratically divergent mass at one loop. By considering first the littlest higgs model we show that this requires a fine tuning: the couplings in the model introduced to give the top quark a mass do not naturally respect the collective symmetry. We show the problem is generic: it arises from the fact that the would be collective symmetry of any one top quark mass term is broken by gauge interactions.

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Contents

1 Introduction

Little Higgs (LH) models offer an alternative to the standard model in which no fundamental scalars need be introduced (for reviews see [1]). Generally, in LH models the Higgs is a composite particle, bound by interactions that become strong at a scale Λ. The mass of the Higgs is much less than Λ as the Higgs is a pseudo-Goldstone boson (PGB) of broken global symmetries in the theory of the new strong interaction.

The global "flavor" symmetry G_f of these models has a subgroup G_w that is weakly gauged. In the absence of this weak gauge force, the flavor symmetry is broken spontaneously to a subgroup H due to hyper-strong interactions at the scale Λ. As a result, there are massless Goldstone bosons that are coordinates on the G_f/H coset space. Since the weakly gauged G_w force breaks the flavor symmetry explicitly, including its effects leads to some of the Goldstone bosons (the would-be Goldstone bosons) being eaten by the Higgs mechanism and the rest becoming PGBs acquiring small masses of order Λ times a small symmetry breaking parameter, the weak gauge coupling constant. The Higgs is the lightest PGB in LH models, and its mass is naturally much less than Λ (and the other PGBs): due to the collective symmetry breaking mechanism its mass arises only at two loops.

Additional interactions must be included in LH models to account for quark and lepton masses. At low energies they reproduce the Yukawa couplings of the standard model. Since these interactions also break the flavor symmetry, they contribute to the masses of the PGBs. In order to ensure that the Higgs remains much lighter than Λ, the quark interactions are designed to implement the collective symmetry breaking mechanism again.

In the littlest higgs [2] model (L^2H) and variants a collective symmetry arises naturally in the gauge sector. Turning off some gauge couplings gives an enlarged symmetry group of the Lagrangian. This ensures that when those couplings are turned off the higgs remains an exact Goldstone boson. The implementation of the collective symmetry in the Yukawa couplings that give rise to the top quark mass is somewhat different. For example, in the L^2H the top-quark doublet is combined with a new quark singlet into a 'triplet' of collective SU(3) symmetry. However, this is not automatically a symmetry of the Lagrangian, as it is not respected by gauge interactions. In other words, the couplings of the doublet and the singlet are a priori independent, but need to be equal in order to implement the collective symmetry mechanism. In this paper we investigate whether it is possible to construct a littlest higgs model for which collective symmetry in the top-quark sector arises naturally.

If one used generic couplings in the L^2H model for the doublet and the singlet of the top quark 'triplet', then loop diagrams induce unsuppressed, order $\Lambda/4\pi$, Higgs masses. Such generic couplings are not forbidden by any symmetry. As we will see, even if the coupling of the 'triplet' is taken to respect the collective SU(3) symmetry at tree level, radiative effects split it into two terms. These, in fact, have different anomalous dimensions (they 'run' differently). The reader can view enforcing collective symmetry in the top quark sector as a fine tuning. Alternatively, one may argue that assuming the symmetry is consistent with the littlest higgs approach. Only an explicit UV completion can validate one view over the other. We will not investigate UV completions in this work, but enquire whether specific models avoid the issue.

Cannot collective symmetry arise naturally by gauging it? After all, if the symmetry is gauged then the restricted form of the quark coupling is a result of the gauge symmetry. For example one may construct a model based on $G_f/H = \mathrm{U}(7)/O(7)$ with $G_w = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^3$. The vacuum aligns [3] so that G_w breaks to the electroweak subgroup $\mathrm{SU}(2) \times \mathrm{U}(1)$ at the scale Λ, and the spectrum has a light Higgs doublet plus many heavier PGBs. Could the gauged SU(3) now play the role of the collective symmetry for the top quark mass? The problem with this is that the gauge symmetry is broken and the would be higgs is eaten. This model is higgsless. This is also generic: the collective symmetry must act nonlinearly on the higgs, and therefore it must be broken. Gauging it eats away the higgs.

In section 2 we review and explain the problem in the L^2H model. The L^2H itself is phenomenologically disfavoured [5] by EWPD, and it is for this reason that alternatives, like models with custodial symmetry [6] or with T-parity [7], have been introduced. Rather than investigating these models individually we show in section 3 that the problem is generic. We first give a very explicit proof for models with $\mathrm{SU}(N)/\mathrm{SO}(N)$ (and $\mathrm{SU}(N)/\mathrm{Sp}(N)$) vacuum manifold. We then generalize, which does not require much additional work. A brief recap is in section 4.

2 Top-quark coupling fine tuning in the Littlest Higgs Model

2.1 Model review

To establish notation we briefly review elements of the L^2H [2]. It has $G_f = \mathrm{SU}(5)$, $H = \mathrm{SO}(5)$ and $G_w = \prod_{i=1,2} \mathrm{SU}(2)_i \times \mathrm{U}(1)_i$. Symmetry breaking $\mathrm{SU}(5) \to \mathrm{SO}(5)$ is characterized by the Goldstone boson decay constant f. The embedding of G_w in G_f is fixed by taking the generators of $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$ to be

$$Q_1^a = \begin{pmatrix} \frac{1}{2}\tau^a & 0_{2\times 3} \\ 0_{3\times 2} & 0_{3\times 3} \end{pmatrix} \qquad \text{and} \qquad Q_2^a = \begin{pmatrix} 0_{3\times 3} & 0_{3\times 2} \\ 0_{2\times 3} & -\frac{1}{2}\tau^{a*} \end{pmatrix} \tag{2.1}$$

and the generators of the $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$

$$Y_1 = \frac{1}{10}\mathrm{diag}(3,3,-2,-2,-2) \qquad \text{and} \qquad Y_2 = \frac{1}{10}\mathrm{diag}(2,2,2,-3,-3). \tag{2.2}$$

The vacuum manifold is characterized by a unitary, symmetric 5×5 matrix Σ. We denote by g_i (g_i') the gauge couplings associated with $\mathrm{SU}(2)_i$ ($\mathrm{U}(1)_i$). If one sets $g_1 = g_1' = 0$ the model has an exact global SU(3) symmetry (acting on the upper 3×3 block of Σ), while for $g_2 = g_2' = 0$ it has a different exact global SU(3) symmetry (acting on the lower 3×3 block). Either of these exact global SU(3) would-be symmetries guarantee the Higgs remains exactly massless. Hence, the Higgs mass should vanish for either $g_1 = g_1' = 0$ or $g_2 = g_2' = 0$. The perturbative quadratically divergent correction to the Higgs mass must be polynomial in the couplings and can involve only one of the couplings at one loop order. Hence it must vanish at one loop. This is the collective symmetry mechanism that ensures the absence of 1-loop quadratic divergences in the higgs mass.

It is standard to introduce the top quark so that the collective symmetry argument still applies. The third generation doublet q_L is a doublet under $\mathrm{SU}(2)_1$ and a singlet under $\mathrm{SU}(2)_2$. Introduce additional $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$-singlet spinor fields: q_R, u_L and u_R. The third generation right handed singlet is a linear combination of u_R and q_R. The charges of these under $\mathrm{U}(1)\times\mathrm{U}(1)$ are listed below, in (2.15). Their couplings are taken to be

$$\mathcal{L}_{\mathrm{top}} = -\frac{1}{2}\lambda_1 f\,\bar{\chi}_{Li}\,\epsilon^{ijk}\,\epsilon^{xy}\,\Sigma_{jx}\,\Sigma_{ky}\,q_R - \lambda_2 f\,\bar{u}_L\,u_R + \mathrm{h.c.} \tag{2.3}$$

where the indexes i, j, k run over 1,2,3, the indexes x, y over 4, 5 and the triplet χ_L is

$$\chi_L = \begin{pmatrix} i\tau^2 q_L \\ u_L \end{pmatrix}. \tag{2.4}$$

The collective symmetry argument now runs as follows. If $\lambda_2 = 0$ then $\mathcal{L}_{\mathrm{top}}$ in (2.3) is constructed so that it exhibits an explicit global SU(3) symmetry, a subgroup of $G_f = \mathrm{SU}(5)$. Under this, the fields χ_L in (2.4) and Σ_{ix} transform as triplets (on $i = 1,2,3$). Since this would-be exact global symmetry is spontaneously broken it guarantees that the Higgs field remains an exactly massless Goldstone boson. Similarly, if $\lambda_1 = 0$ then there is no coupling of the quarks to the Goldstone bosons, which therefore remain massless. Hence, the mass term must vanish as either λ_1 or λ_2 are set to zero, and since the quadratic divergence is polynomial in the couplings, it can only arise at two loops.

The gauge and top-quark interactions generate an effective, Coleman-Weinberg potential which determines the vacuum orientation. If the gauge couplings are strong enough [8],

$$g_1'^2 + g_1^2 > \frac{2N_c}{3\pi^2 c}\,\lambda_1^2\,\lambda_2^2\left[\ln\left(\frac{\Lambda^2}{(\lambda_1^2+\lambda_2^2)f^2}\right) + \frac{\hat{c}'}{2}\right]. \tag{2.5}$$

where c and $\hat{c}'$ are unknown dynamical constants of order unity, the vacuum alignment is

$$\Sigma_{ew} = \begin{pmatrix} 0 & 0 & \mathbf{1}_{2\times 2} \\ 0 & 1 & 0 \\ \mathbf{1}_{2\times 2} & 0 & 0 \end{pmatrix}. \tag{2.6}$$

leading to the gauge-symmetry breaking into the electroweak subgroup, $\prod_{i=1,2} \mathrm{SU}(2)_i \times \mathrm{U}(1)_i \to \mathrm{SU}(2) \times \mathrm{U}(1)$.

2.2 The hidden fine tuning

As we just saw, the top quark Lagrangian $\mathcal{L}_{\text{top}}$ in (2.3) is constructed so that it exhibits an explicit global SU(3) symmetry. However, this is a symmetry of the Lagrangian only for $\lambda_2 = g_1 = g_1' = 0$.

There is in fact no symmetry reason for the fields in χ_L to combine into a triplet. Given that the effective Lagrangian is restricted only by the non-linear realization of the symmetry (by parametrizing G_f/H) and by the requirement of explicit gauge invariance under G_w, the coupling in (2.3) is more generally of the form

$$\mathcal{L}_{\text{top}} = -\lambda_1 f \bar{q}_L^{\,i} \epsilon^{xy} \Sigma_{ix} \Sigma_{3y} q_R - \frac{1}{2} \lambda_1' f \bar{u}_L \epsilon^{3jk} \epsilon^{xy} \Sigma_{jx} \Sigma_{ky} q_R - \lambda_2 f \bar{u}_L u_R + \text{h.c.} \tag{2.7}$$

Only when $\lambda_1' = \lambda_1$ (and $\lambda_2 = g_1 = g_1' = 0$) do we recover the global SU(3) symmetry of the collective symmetry mechanism. The main observation of this work is that the relation $\lambda_1' = \lambda_1$, assumed throughout the little higgs literature, is unnatural. We refer to this as the hidden fine tuning problem. The reader may choose not to see this as a problem, that assuming collective symmetry in the Yukawa sector at tree level is in line with the littlest higgs approach. Only an explicit UV completion can validate one view over the other.

Although $\lambda_1' = \lambda_1$ is natural in the absence of the gauge interactions, these are already present in the UV completion. Below we comment in slightly more detail on how radiative effects explicitly introduce SU(3) breaking into the Yukawa couplings.

It should be evident that for $\lambda_1' \neq \lambda_1$ the collective symmetry argument is spoiled. A straightforward computation gives a quadratically divergent correction to the higgs mass,

$$\delta m_h^2 = \frac{12}{16\pi^2} (\lambda_1^2 - \lambda_1'^2) \Lambda^2 \tag{2.8}$$

where Λ is a UV cut-off. The severity of the fine tuning can now be explored. If we insist that the Higgs mass should be naturally of order of 100 GeV, while $\Lambda \sim 10$ TeV, then, not surprisingly, $\lambda_1' - \lambda_1 \lesssim (4\pi m_h/\Lambda)^2 \sim 1\%$.

The Lagrangian in (2.7) is not the most general one consistent with symmetries to lowest order in the chiral expansion. If SU(3) were a good symmetry one could add to the Lagrangian a term of the form

$$\bar{\chi}_{Li} \epsilon_{jkl} \epsilon_{xy} (\Sigma^*)^{ij} (\Sigma^*)^{kx} (\Sigma^*)^{ly} q_R \tag{2.9}$$

One can also freely replace $q_R \leftrightarrow u_R$ in eqs. (2.3) and (2.9), and then, of course, split each SU(3) invariant term into a sum of SU(2) $\times$ U(1) invariant terms. There is no reason a priori why these terms should be ignored, but they are not dangerous. In fact, they are inevitable, as they are generated radiatively, many of them already at one loop [9].

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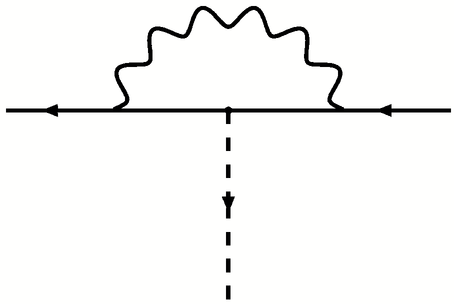

Figure 1. Feynman diagram that contributes to the renormalization of the Yukawa couplings λ_1 and λ_1'. The wavy line represents a gauge boson of $U(1)_1$ and the solid and doted lines a spinor and a PGB, respectively.

2.3 Radiatively induced $\lambda_1' \neq \lambda_1$

Imposing $\lambda_1' - \lambda_1 = 0$ is not only a fine tuning, it is unnatural. Since the symmetry is broken by marginal operators, the renormalization group evolution of the difference $\lambda_1' - \lambda_1$ takes it away from zero, even if it is chosen to be zero at some arbitrary renormalization point μ.

As a check we have computed explicitly the one loop renormalization group equations for these couplings (see figure 1):

$$\mu \frac{\partial}{\partial \mu} \ln \left(\frac{\lambda_1}{\lambda_1'} \right) = \left(\tfrac{2}{3} - y \right) \frac{3 g_1'^2}{16\pi^2} \tag{2.10}$$

Here y is the charge of q_R under $U(1)_2$. Details of the calculation will be presented elsewhere [9]. If $\beta_{g_1'} = (b/16\pi^2) g_1'^3$ then we can write the solution in terms of the running coupling:

$$\frac{\lambda_1(\mu)}{\lambda_1'(\mu)} = \frac{\lambda_1(\Lambda)}{\lambda_1'(\Lambda)} \left(\frac{g_1'(\mu)}{g_1'(\Lambda)} \right)^{\frac{2-3y}{b}} \tag{2.11}$$

The numerical value for b can be obtained from the standard QED beta function (see [10])

$$b = \frac{2}{3} \sum_{\text{Weyl fermion}} Y_{1i}^2 + \frac{1}{6} \sum_{\text{real scalar}} Y_{1i}^2 \tag{2.12}$$

To compute this, we need to introduce the Yukawa-type coupling for all the other standard model quarks. We will follow Perelstein [1] by noting that there is no need for implementing collective symmetry breaking for the other standard model quarks due to their small Yukawa couplings. Thus the other "up" type quarks Yukawa interaction can be introduced by

$$-\lambda_\alpha^u f \bar{q}_{\alpha L}^{\,i} \epsilon^{xy} \Sigma_{ix} \Sigma_{3y} q_{\alpha R} \tag{2.13}$$

where $\alpha = 1, 2$ is the quark family index. Similarly the other "down" type quark interactions can be introduced by

$$-\lambda_\alpha^d f \bar{q}_{\alpha L}^{\,i} \epsilon_{xy} (\Sigma^*)^{ix} (\Sigma^*)^{3y} d_{\alpha R} \tag{2.14}$$

here $\alpha = 1,2,3$. If we take $Y_2(q_R) = y$, then the Y_1 charge of all the particles involved are

$$\begin{array}{|c|ccccccc|}\hline & q_{\alpha L} & q_{\alpha R} & d_{\alpha R} & u_L & u_R & H & \phi \\ \hline Y_1 & \frac{11}{30}-y & \frac{2}{3}-y & \frac{1}{15}-y & \frac{13}{15}-y & \frac{13}{15}-y & 1/4 & 1/2 \\ Y_2 & y-\frac{1}{5} & y & y-\frac{2}{5} & y-\frac{1}{5} & y-\frac{1}{5} & 1/4 & 1/2 \\ \hline\end{array} \tag{2.15}$$

Thus we get $b = \frac{1}{360}\left(2737 - 8832y + 10080y^2\right) \geq 46/105$. However, we note that the y can be arbitrary.

We do not dwell on the numerics, since there are too many adjustable parameters (the choice of y, the value of U(1) couplings and $\lambda_{1,2}(\Lambda)$ which however must satisfy (2.5), the value of the cutoff Λ). We simply note that $1/16\pi^2 \log(\Lambda/m_h) \sim 1/16\pi^2 \log(100) \sim 3\%$. Hence, even fine tuning $\lambda_1(\Lambda) = \lambda_1'(\Lambda)$ generically produces a difference $\lambda_1(m_h) - \lambda_1'(m_h)$ in excess of 1%.

Note also that the same behavior must occur in the UV completion of the L^2H model. After all, the terms in the Lagrangian that break G_f-symmetry model the effects of symmetry breaking interactions at short distances, that is, in the UV completion. The interactions in the UV completion that are responsible for the quark Yukawa couplings cannot be taken to respect the SU(3) symmetry required for the collective symmetry argument. The breaking of the SU(3) symmetry in the UV completion is naturally much larger than in (2.11) since neither the $U(1)_1$ gauge coupling nor the Yukawa couplings are asymptotically free.

3 A no-go theorem

In this section we show the impossibility of constructing a theory that implements without fine tuning the collective symmetry mechanism on the terms responsible for quark and lepton masses. Let us begin by stating in general terms what is required in order to implement the collective symmetry mechanism. Any given term in the Lagrangian has to be symmetric under a subgroup G_c of the flavor group G_f under which the higgs field transforms non-linearly, and in particular, with a transformation that includes a constant shift.[1] In addition, there must not be any one loop divergent radiative corrections that involve the coupling constants for two different terms.

Of course there are additional requirements on each individual term in the Lagrangian. In particular any one term must be invariant under G_w, the gauged subgroup of G_f. We do not wish to specify this gauge group, since one could look for realizations of the collective symmetry mechanism in gauge groups other than the one of the L^2H. Below we will only need to use the fact that this group contains the electroweak gauge group, $G_{\rm ew} = SU(2) \times U(1)$, that this symmetry is linearly realized, i.e., that $G_{\rm ew} \subset H$ so it remains unbroken at the scale at which G_f breaks to H, and that the higgs field must transform as a doublet with hypercharge 1/2 under the electroweak group.

The hidden fine tuning problem in the quark sector of the L^2H resulted from the fact that $G_c = SU(3)$ is not a symmetry of the Yukawa term, because G_c does not commute

[1] Different terms in the Lagrangian may be invariant under different collective symmetry groups G_c.

with G_w. The Yukawa term in the Lagrangian is actually a sum of terms that are separately invariant under the gauge group and the collection of terms can only be symmetric under G_c by fine tuning the separate coupling constants at one scale. There are two ways that immediately come to mind in which one could try to extend the L^{2}H model to get around this problem. Either extend the gauge group so that G_c itself is gauged or obtain G_c as an accidental symmetry. These, or other strategies cannot work: below we will prove in generality that the collective symmetry mechanism cannot work for terms other than the kinetic terms in the Lagrangian.

3.1 An SU(7)/SO(7) example and its generalization to SU(N)/SO(N)

It is simpler to understand the general case by first looking at an explicit example. We can motivate this by the following observation. If the SU(3) collective symmetry that acts on the first three rows and columns of Σ is elevated to a gauge symmetry, then the equality $\lambda_1' = \lambda_1$ is natural. Of course, in the L^{2}H model this won't work because the SU(3) is broken at the scale Λ at which SU(5) breaks to SO(5), and the higgs is eaten at this scale. But perhaps one can construct a theory based on a larger G_f symmetry group with SU(3) gauged and the higgs still transforming non-linearly under some G_c subgroup of G_f.

For example, one may consider a nonlinear sigma model based on $G_f/H = \mathrm{U}(7)/O(7)$ (with spinor fields in non-trivial representations of the hyper-strong gauge group so that the U(1) in $\mathrm{U}(7) = \mathrm{SU}(7) \times \mathrm{U}(1)$ is non-anomalous). Assume the U(7) is broken to $O(7)$ by a symmetric condensate, which transforms under U(7) as $\Sigma \to V\Sigma V^T$. Now gauge a $G_w = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^3$ subgroup of U(7). The SU(3) factor is precisely the gauged version of the top-block collective symmetry group, under which the royal triplet χ_L transforms as an actual triplet. It is a straightforward, if lengthy, exercise to show that the vacuum aligns correctly, that is, G_w breaks to the electroweak subgroup. One can identify Π_{i4}, and related entries, with the higgs doublet. By suitably choosing the generators of the gauged $\mathrm{U}(1)^3$ symmetry one finds that the higgs field is the only light PGB.

Now introduce top quark couplings in a manner consistent with the collective symmetry and without fine tuning of Yukawa couplings. Just as in the L^{2}H model, in addition to the third generation quark doublet q_L and singlet q_R, introduce a pair of weak singlet Weyl fermions u_L and u_R that transform as $\mathbf{1}_{1/6}$ under $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$. The singlet u_L is combined with the doublet q_L into a triplet of the gauged SU(3), precisely as in (2.4). By suitably choosing the transformation properties under the $\mathrm{U}(1)^3$ we can ensure that the most general Yukawa Lagrangian consistent with the symmetries, to lowest order in the chiral expansion, is

$$\mathcal{L}_{\mathrm{top}} = -f\lambda_1 \bar{\chi}_{Li} (\Sigma^*)^{i4} q_R - \frac{1}{2} f \lambda_2 \bar{\chi}_{Li} \epsilon^{ijk} \epsilon^{xy} \Sigma_{jx} \Sigma_{ky} u_R + \mathrm{h.c.} \tag{3.1}$$

where the indexes i, j, k run over $1, 2, 3$ and x, y over $5, 6$. The problem with this model is that the SU(3) symmetry does not protect the higgs. The collective symmetry required is an SU(4) acting on the top-left 4×4 block of Σ. This in turn requires enlarging the true triplet to a four-plet, which allows for more terms in the Lagrangian, which are related by the U(4) symmetry. However, this is not a good symmetry of the Lagrangian and the added terms are related to the ones above only by imposing unnaturally a collective symmetry. This is precisely the same problem we encountered with the L^{2}H.

Let us generalize this to models with SU(N)/SO(N) vacuum manifold, parametrized by the $N \times N$ symmetric unitary matrix Σ. We assume there is an SU(2) $\times$ U(1) gauged subgroup of SO(N). Without loss of generality we can take its embedding in SU(N) as follows:

$$Q^a = \frac{1}{2}\begin{pmatrix} \tau^a & 0_{2\times(N-4)} & 0_{2\times 2} \\ 0_{(N-4)\times 2} & 0_{(N-4)\times(N-4)} & 0_{(N-4)\times 2} \\ 0_{2\times 2} & 0_{2\times(N-4)} & -\tau^{a*} \end{pmatrix}$$
$$Y = \frac{1}{2}\mathrm{diag}(1, 1, y_3, \ldots, y_{N-2}, -1, -1) \tag{3.2}$$

with $\sum y_i = 0$. We assume further that the whatever other interactions exist they align the vacuum along (2.6) (with the proper interpretation for the dimensions of the 0 blocks and the center unit block). Then, as usual, $\Sigma = \exp(i\Pi/f)\Sigma_{ew}\exp(i\Pi^T/f) = \exp(2i\Pi/f)\Sigma_{ew}$, where in the last step we have chosen the broken generators to satisfy $\Pi\Sigma_{ew} = \Sigma_{ew}\Pi^T$. The $N-4$ doublets Π_{ix} with $i = 1, 2$ and $x = 3, \ldots, N-2$, have hypercharge $1/2 + y_x$. So any one of these for which $y_x = 0$ is a prospective higgs doublet.

Under an infinitesimal SU(N) transformation, $1 + i\epsilon^a T^a$, the matrix of goldstone bosons transforms as

$$\delta\Pi = \frac{f}{2}(T^a + \Sigma_{ew}T^{aT}\Sigma_{ew}^\dagger) + \cdots \tag{3.3}$$

where the ellipses stand for terms at least linear in the fields. We are interested in finding a subgroup G_c of SU(N) under which the higgs field transformation includes a constant shift. However any such transformation does not commute with SU(2) $\times$ U(1). Without loss of generality we assume that the third entry has zero hypercharge, $y_3 = 0$, so that $\Pi_{i3} = \Pi^*_{3i} = \Pi_{(N-2)i} = \Pi^*_{i(N-2)}$ is the prospective higgs doublet. Then G_c must contain generators

$$X = \begin{pmatrix} 0_{2\times 2} & x_{2\times 1} & 0_{2\times(N-3)} \\ x^\dagger_{1\times 2} & 0_{1\times 1} & 0_{1\times(N-3)} \\ 0_{(N-3)\times 2} & 0_{(N-3)\times 1} & 0_{(N-3)\times(N-3)} \end{pmatrix} \tag{3.4}$$

or

$$X = \begin{pmatrix} 0_{(N-3)\times(N-3)} & 0_{(N-3)\times 1} & 0_{(N-3)\times 2} \\ 0_{1\times(N-3)} & 0_{1\times 1} & x^T_{1\times 2} \\ 0_{2\times(N-3)} & x^*_{2\times 1} & 0_{2\times 2} \end{pmatrix} \tag{3.5}$$

with x a complex two component vector. Both of these give the same linear shift on the prospective higgs field, as can be verified by computing $X + \Sigma_{ew}X^T\Sigma_{ew}^\dagger$. It follows that for either one of these generators we have

$$[Q^a, X] = X' \tag{3.6}$$

where X' is a generator of the form of X. This means that the X generators transform under SU(2) as a tensor operator; they are in fact complex doublets with hypercharge 1/2, just like the higgs. Now, there are additional generators in G_c: at the very least it contains the SU(3) subgroup generated by the top-left or bottom right 3×3 blocks. Together, X and these additional generators transform as a *reducible* representation of the electroweak subgroup. It follows that a gauge invariant term in the Lagrangian that is also invariant under G_c is a sum of terms that are individually gauge invariant. The only exception is when the term is constructed of fields that are separately SU(2) invariant, as is the case of the λ_2 mass term, in (2.3), in the L^{2}H model. But it is unnatural to choose the coefficients of these various terms to make their sum G_c invariant. This is because the gauge interactions always break the symmetry. Gauge boson exchange Feynman diagrams like that of figure 1 give divergent corrections to these couplings, and the corrections do not preserve the G_c invariance.

We can relax one assumption above slightly. We do not need to assume the vacuum alignment Σ_{ew} is along (2.6). In order to have a collective symmetry argument that one can already apply in the gauge sector one needs the first and last two rows and columns to be as in (2.6). But the central $(N-4) \times (N-4)$ block does not have to be a diagonal matrix, only a unitary, symmetric matrix. However, the argument goes through as before: the components of Π that we identify with the higgs are changed in precisely the way that the shifts in (3.3) are modified and the rest of the argument goes through unchanged.

The explicit proof for the case $G_f/H = \mathrm{SU}(N)/\mathrm{Sp}(N)$ is completely analogous.

3.2 The general case

We turn now to the general case. We assume that G_w contains the electroweak gauge group $G_{\mathrm{ew}} = \mathrm{SU}(2) \times \mathrm{U}(1)$, with $G_{ew} \subset H$. We further assume that a subset of goldstone bosons can be identified with the higgs field. We consider a term in the Lagrangian that is both symmetric under G_{ew} and has a collective symmetry G_c. We show in the appendix that we only need to consider semi-simple G_c, which we assume henceforth.

That the higgs transforms linearly under the electroweak gauge group means that there is a doublet h in Π that transforms as

$$\delta_\epsilon h = i\epsilon^a \frac{\tau^a}{2} h + i\epsilon \frac{1}{2} h \tag{3.7}$$

under $\mathrm{SU}(2) \times \mathrm{U}(1)$. Under a group $G_c \in G_f$ h transforms non-linearly,

$$\delta_\eta h = \eta^m x^m + \cdots \tag{3.8}$$

where the implicit sum over m is over all generators in G_c, for some two component complex vectors x^m and the ellipses stand for terms at least linear in h. One can redefine the basis of generators in G_c so that $x^m = 0$ for $m \geq 5$ and x^m for $m = 1, \cdots, 4$ are unit vectors, with $m = 1, 3$ real and $m = 2, 4$ purely imaginary. Now consider the commutator,

$$(\delta_\eta \delta_\epsilon - \delta_\epsilon \delta_\eta) h = i\epsilon^a \eta^m \frac{\tau^a}{2} x^m + i\epsilon \eta^m \frac{1}{2} x^m + \cdots \tag{3.9}$$

The commutator is again a non-linear transformation, a linear combination of the same four generators in G_c that shift the higgs. In terms of the Lie algebra of G_f, denoting these generators by X^i, with[2] $i = 1, 2$ and the generators of $G_{\rm ew}$ by Q^a and Y, we read off

$$[Q^a, X^i] = \frac{i}{2}(\tau^a)^{ij} X^j, \qquad [Y, X^i] = \frac{i}{2} X^i \tag{3.10}$$

This is precisely the statement in eq. (3.6), derived there from the explicit form of matrices, that the generators transform as tensors of $G_{\rm ew}$ with the same quantum numbers as the higgs doublet, but we see now that it holds more generally, independently of those explicit matrix representations.

Since there is no semi-simple Lie algebra of rank 4, there must be additional generators, and $[X^i, X^j]$ must give some of these additional generators. Denote a non-vanishing commutator by $\hat{X}^{ij} = [X^i, X^j]$. Using the Jacobi identity we see that

$$[Q^a, \hat{X}^{ij}] = [Q^a, [X^i, X^j]] \tag{3.11}$$

$$= [X^i, [Q^a, X^j]] - [X^j, [Q^a, X^i]] \tag{3.12}$$

$$= \frac{i}{2}(\sigma^a)^{jk} X^{ik} - \frac{i}{2}(\sigma^a)^{ik} X^{jk} \tag{3.13}$$

So these generators also satisfy an equation like (3.6) but transform in a representation in the tensor product of two doublets. Continuing this way, considering commutators of the generators we have so far, we can eventually generate the complete Lie algebra and find that it breaks into sectors classified by irreducible representations under $G_{\rm ew}$.

We can use this to show that invariants under G_c break into a sum of terms separately invariant under $G_{\rm ew}$. Any non-trivial invariant must be a product of two combination of fields, one transforming in some irreducible representation R of G_c and the other as the complex conjugate $\bar{R}$. But from the previous paragraph it follows that under $G_{\rm ew}$ the representation R breaks into a direct sum $R = r_1 \oplus r_2 \oplus \cdots$ of at least two irreducible representations of $G_{\rm ew}$. Therefore the product $R \times \bar{R}$, contains the sum of at least two invariants under $G_{\rm ew}$, $r_1 \times \bar{r}_1$ and $r_2 \times \bar{r}_2$. Since G_c is not a symmetry of the theory (because the kinetic energy term for the goldstone bosons is not invariant), the two (or more) $G_{\rm ew}$ invariants can be summed into a G_c invariant only by fine tuning coefficients in the Lagrangian. This completes the argument.

It may not be self-evident that any non-trivial representation of G_c breaks into two or more representations under $G_{\rm ew}$. This can be shown by noting that the roots of the Lie algebra, that is the weights of the adjoint representation, of G_c break into a sum of irreducible representations of $G_{\rm ew}$, precisely the same representations that the generators fall into.[3] Then by following the same procedure as in establishing branching rules for representations of Lie algebras, that is, introducing projection operators in weight space, and using the fact that the roots form irreducible representations, one obtains that every representation of G_c is decomposed into a sum of irreducible representations of $G_{\rm ew}$.

[2]The index i runs over 1,2 because the hermitian matrices break into a symmetric and an antisymmetric part, corresponding to the two real and two imaginary components of x^m, and also to the real and imagnary components of the higgs doublet.

[3]This follows form considering the standard map $T^A \to |T^A\rangle$ of the generators of G_f, with $T^A|T^B\rangle = |[T^A, T^B]\rangle$. Then $Q^a|X^i\rangle = i/2(\sigma^a)^{ij}|X^j\rangle$ and so on for the other generators of G_c.

We remarked above that U(1) factors in G_c are ignored. This requires some explanation. After all, one could conceivably take the four broken generators to generate a collective symmetry group of dimension 4, say $U(1)^4$ or $SU(2) \times U(1)$. But the U(1) symmetries do not help insure the higgs remains massless. It is easy to see why by considering first the familiar L^{2}H case. The λ_1 and λ_1' terms of (2.7) that need to be related by collective symmetry to obtain necessary cancellations in one loop graphs can be made separately invariant under several U(1) symmetries. In fact, the situation is reversed from the semi-simple group case, where a representation R of G_c is a direct sum of at least two irreducible representation of G_{ew}. Since the irreducible representations of U(1) are one dimensional, it is G_{ew} that relates several irreducible representation of U(1), and forces them together into a term in the Lagrangian.

4 Conclusions

It is easy to see that radiative effects break the collective symmetry of top quark couplings of the L^{2}H model. These effects must also be present in the underlying UV completion so they cannot be dismissed as small. Also, they are generically too large for successful phenomenology even if one chooses to enforce collective symmetry on the tree level top Yukawa couplings.

The problem cannot be circumvented by enlarging the model to one with a larger underlying flavor symmetry group. Gauging collective symmetry is not an option: it either gives a higgsless model or again requires imposing an unnatural symmetry at tree level to avoid quadratically divergent radiative corrections to the higgs mass. The reader may see this as a fine tuning problem, or may adopt the view that imposing the collective symmetry on the top quark sector is in keeping with the littlest higgs strategy.

We have shown that the collective symmetry argument cannot be implemented naturally on the Yukawa couplings of little higgs models. Of course, no-go theorems are only as good as its assumptions. We did not prove that no model exists that can both include top quarks and solve the little hierarchy problem. For example, one can presumably partially supersymmetrize the model to ensure the cancellation of top loop induced quadratic mass divergences, at least at one loop. In the absence of a novel mechanism to suppress the quadratic divergences in the top quark-induced radiative corrections to the higgs mass without fine tuning, it seems one must rely on a UV completion to explain the approximate collective symmetry of the model.

A Proof that G_c is semi-simple

We show that the four generators X^i in G_c that produce the non-linear transformations of the four real components of the higgs field are in a subalgebra that generates a semi-simple subgroup of G_c.

Our starting point are the commutation relations

$$[Q^a, X^i] = R(Q^a)^{ij} X^j, \qquad [Y, X^i] = R(Y)^{ij} X^j. \tag{A.1}$$

These are part of the algebra of G_f. We recall some basic facts about compact Lie algebras (we follow and use the notation of ref. [11]). The Cartan subalgebra of G_f is the largest set of mutually commuting generators H_i, $i = 1, \ldots, r \equiv \text{rank}(G_f)$. In the adjoint representation define a vector space by the map $T^A \to |T^A\rangle$, where the T^A are generators of G_f, and define the action of generators on this vectors by $T^A|T^B\rangle = |[T^A, T^B]\rangle$. Moreover, define an inner product on this space by $\langle T^A|T^B\rangle = \text{Tr}(T^{A\dagger}T^B)$. Since the H_i are mutually commuting one can find a basis of the vector space $H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle$. The states correspond to the rest of the generators, E_α. It follows that $[H_i, E_\alpha] = \alpha_i E_\alpha$ and $E_{-\alpha} = E_\alpha^\dagger$. Choose the generators of the Cartan subalgebra to satisfy $\langle H_i|H_j\rangle = \delta_{ij}$. It can be shown

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^{r} \alpha_i H_i \tag{A.2}$$

We intend to show that there is a basis of of Cartan generators for which the four X^i (or a linear combination of them) correspond to two pairs $(E_\alpha, E_{-\alpha})$ that therefore do not commute among themselves.

We are free to take $H_1 = Q^3$ and $H_2 = Y$ as the first two members of the Cartan subalgebra. Now, the standard representation

$$R(Y) = \begin{pmatrix} -\tau^2/2 & 0 \\ 0 & -\tau^2/2 \end{pmatrix}, \qquad R(Q^3) = \begin{pmatrix} -\tau^2/2 & 0 \\ 0 & \tau^2/2 \end{pmatrix}, \tag{A.3}$$

$$R(Q^1) = \begin{pmatrix} 0 & -\tau^2/2 \\ -\tau^2/2 & 0 \end{pmatrix}, \qquad R(Q^2) = \begin{pmatrix} 0 & -i/2\, \mathbb{1}_{2\times 2} \\ i/2\, \mathbb{1}_{2\times 2} & 0 \end{pmatrix}, \tag{A.4}$$

and we can make a transformation $X^i \to U^{ij} X^j$ to diagonalize $R(Q^3)$ and $R(Y)$:

$$\begin{aligned}
[Y, X^1] &= -\frac{1}{2}X^1 & \qquad [Q^3, X^1] &= -\frac{1}{2}X^1 \\
[Y, X^2] &= +\frac{1}{2}X^2 & \qquad [Q^3, X^2] &= +\frac{1}{2}X^2 \\
[Y, X^3] &= -\frac{1}{2}X^3 & \qquad [Q^3, X^3] &= +\frac{1}{2}X^3 \\
[Y, X^4] &= +\frac{1}{2}X^4 & \qquad [Q^3, X^4] &= -\frac{1}{2}X^4
\end{aligned} \tag{A.5}$$

The rest of the Cartan subalgebra can be chosen to commute with X^i, as we now show. Suppose

$$\begin{aligned}
[H_i, X^1] &= -\frac{a_i}{2}X^1 \\
[H_i, X^2] &= +\frac{a_i}{2}X^2 \\
[H_i, X^3] &= -\frac{b_i}{2}X^3 \\
[H_i, X^4] &= +\frac{b_i}{2}X^4
\end{aligned} \tag{A.6}$$

Then the generators $H'_i = H_i - (a_i + b_i)/2\, Y - (a_i - b_i)/2\, Q^3$, commute with X^j.

Now, it is clear the the four $|X^i\rangle$ are among the states $|E_{\alpha'}\rangle$ that satisfy $H'_i|E_{\alpha'}\rangle = \alpha'_i|E_{\alpha'}\rangle$. Moreover the vectors α' for X^i are of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, 0, \ldots, 0)$. Equation (A.2) holds only provided the H_i that satisfy $\mathrm{Tr}(H_i H_j) = \delta^{ij}$. While the H_i satisfy this orthonormality condition, the new basis H'_i generally does not. Writing $H_i = V_{ij}H'_j$ gives an explicit set of eigenvectors of H_i,

$$H_i|E_{\alpha'}\rangle = V_{ij}H'_j|E_{\alpha'}\rangle = V_{ij}\alpha'_j|E_{\alpha'}\rangle$$

The eigenvectors are the same as those of H'_i, and hence the $|X^j\rangle$ are still among them, but the eigenvalues have changed, $\alpha_i = V_{ij}\alpha'_j$. But with this basis we can use (A.2). Explicitly

$$[X^1, X^2] = -\frac{1}{2}\sum_{i=1}^{r}(V_{i1} + V_{i2})H_i \tag{A.7}$$

$$[X^3, X^4] = -\frac{1}{2}\sum_{i=1}^{r}(-V_{i1} + V_{i2})H_i \tag{A.8}$$

We see that both commutators are non-vanishing, as we set out to demonstrate.

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